Comment on "Relationship Between Kane's Equations and the Gibbs-Appell Equations"

J. E. Keat*
Photon Research Associates Inc.
Cambridge, Massachusetts

Introduction

DESLOGE'S Note[1] is a continuation of exchanges[2-4] between himself and Kane on the topic. As discussed herein, the Note denegrates Kane's contribution to dynamics, presents highly controversial interpretations of the relation between the two equations, and reaches erroneous conclusions about which requires the least labor in applications.

I regard the one-particle example in the Note as inadequate for discourse on the methods. Therefore, a more general form of the equations is presented. Consider a system S of ν nonrigid bodies Z_i which is modeled with n generalized coordinates q_α and m nonholonomic constraints. Kane's equation for S can be written as

$$\tilde{F}+\tilde{F}^*=0 \tag{1a}$$

where

$$\tilde{F}\triangleq\sum_i \int \tilde{V}_i\cdot \boldsymbol{R}_i \mathrm{d}m \tag{1b}$$

$$-F^*\triangleq\sum_i \int \tilde{V}_i\cdot \boldsymbol{a}_i \mathrm{d}m \tag{1c}$$

with $i=1$-ν. $\tilde{F}$ and $\tilde{F}^*$ are p vectors where $p=n-m$. For simplicity, only the body force vectors $\boldsymbol{R}_i$ are shown in Eq. (1b). V_i is the transpose of the operator in the equation

$$\boldsymbol{v}_i=\tilde{V}_i^T\cdot u+\tilde{\boldsymbol{v}}_{ti} \tag{2}$$

where u contains the generalized speeds. When coordinate frame resolution is introduced, u is a p vector. Then $\boldsymbol{v}_i$ and $\boldsymbol{a}_i$ are the velocity and acceleration vectors, relative to the inertial frame N, of a generic particle of Z_i. The notation is similar to Kane's.[5]

The Gibbs-Appell (G-A) equation for S is

$$G_{\dot{u}}=\tilde{F} \tag{3a}$$

where

$$2G\triangleq\sum_i \int \boldsymbol{a}_i\cdot \boldsymbol{a}_i \mathrm{d}m \tag{3b}$$

The p vector $G_{\dot{u}}$ is the gradient of the scalar function G. The $\boldsymbol{a}_i$ are functions of $\dot{u}$, u, q, and t where q is an n vector. The G-A equation is discussed in texts referenced in the Note and also in Whittaker.[6]

Kane's Equation

Kane's first papers on his method[7,8] appeared in 1961 and 1965. Subsequently, he and his colleagues created a large body of work on its application. In his first papers, he did not give a name to his equation. In his 1968 text[9] he called it "Lagrange's form of D'Alembert's principle." Later, Huston and Passerello[10] and Huston et al.,[11] also used this term when discussing Kane's method. Likins[12] applied the term only to the restricted form of Eq. (1) in which u includes only the time derivatives of real generalized coordinates; he believed the generalization to quasicoordinate time derivatives to be an extension by Kane to this historical result, and he applied the term "Kane's equations" to this more general form. The term "Lagrange's form of D'Alembert's principle" never took root. The method has been well known since the early 70's, and it has always been referred to verbally as "Kane's equation" or "Kane's method." Contrary to Desloge, there is no indication that the result of Ref. 13 was the reason Kane adopted this term himself in later publications.

While engineers are well aware of the existence of Kane's method, they have yet to embrace it passionately. Instead, most dynamicists still prefer classical methods, particularly Lagrange's equation. Lagrange's equation is a powerful tool, but it possesses drawbacks when applied to complicated problems.[14] Foremost among these are its inability to handle quasicoordinate time derivatives and the necessity of deriving and differentiating a complicated expression for kinetic energy. Both these difficulties are eliminated by Kane's method. Unlike Lagrange's equation, Kane's method does not handle nonholonomic constraints by a Lagrange multiplier vector λ. Instead, it handles them by the use of a u whose dimension is less than that of q. However, Kane's method can be altered to handle constraints via λ, if one chooses. We believe that in spite of its slow start, the use of Kane's method will increase dramatically in future decades.

Desloge proposes that Eq. (1) be called Kane's form of the Gibbs-Appell equation or the Gibbs-Appell-Kane equation. These names are inappropriate. Neither Gibbs nor Appell originated the equation, and the name of neither has ever been associated with it. Kane was the first to recognize its usefulness to dynamics problems in the computer age and to spend decades promoting it. Therefore, it should continue to be called Kane's equation.

Relationship Between Kane's Equation and the Gibbs-Appell Equation

A comparison of Eqs. (1) and (3) shows that Kane's equation and the G-A equation are the same basic equation in different garb, as Desloge contends. Thus, as Desloge asserts, Kane's method and the G-A method will yield the same equations of motion in applications. However, other assertions Desloge makes on the relation between the two equations and methods are controversial.

Desloge asserts that "Kane's equations are simply a particular form of the Gibbs-Appell equations." I disagree. The inertia force vector $\tilde{F}^*$ is a fundamental quantity. The expression on the right side of Eq. (1c) and $G_{\dot{u}}$ are different ways of specifying $-\tilde{F}^*$. Therefore, Kane's equation is not a particular form of the G-A equation, and vice versa.

Desloge also asserts that "Kane's method is simply a particular method of applying the Gibbs-Appell equations." Again, I disagree. Few would claim one is employing Lagrange's equation if one derives equations of motion by a procedure that does not literally involve forming T or L and differentiating. The same criterion applies to the G-A equation. Hence, when one derives equations of motion by a procedure that does not literally involve forming G and differentiating it, one is not employing the G-A method. Thus, Kane's method is distinct from the G-A method.

Which Method is Better?

In order to compare the two methods in aerospace applications, I applied Eqs. (1-3) to the following simple problems: 1) single nonrigid body, 2) two rigid bodies connected by a pin joint, and 3) spacecraft deploying a boom. The following conclusions are based on this study.

Received March 30, 1987.

*Senior Scientist.

Desloge considers two methods of differentiating G. The first, G-A(1), uses implicit differentiation. The second, G-A(2), completes the squares before differentiating. He asserts that Kane's method and G-A(1) require the same operations and hence the same labor. This assertion is not precisely true. In practice, G-A(1) requires slightly more labor due to the "overhead" operations of literally writing down the expression for G and literally differentiating it. Also, G-A(1) is more cumbersome conceptually, since it requires the analyst to introduce the ideas of G and differentiation into his thinking. For these reasons, I recommend Kane's method over G-A(1). Why complicate things by introducing G and differentiation into one's thinking and mathematics when they serve no purpose?

Desloge asserts that there may be situations in which G-A(2) is preferable to G-A(1). Possibly, but I cannot think of any. In the simple problems noted earlier, the labor required by G-A(2) was so much greater than that of G-A(1) that I did not finish it. It is evident that in problems of realistic engineering complexity, the labor required by G-A(2) will make it virtually impractical. Therefore, G-A(2) is not a serious competitor to Kane's method.

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Comment on "Relationship Between Kane's Equation and the Gibbs-Appell Equations"

Dan E. Rosenthal*
Symbolic Dynamics, Mountain View, CA

An Engineering Note is intended to communicate new developments. The article by Desloge[1] is virtually identical to a previous article by Desloge.[2] This earlier article should have been referenced in the article under discussion here.

Desloge attempts to make several points purporting to show the superiority of Gibbs' method over Kane's method. He claims that "the Gibbs-Appell equation is more manifestly a generalized equation than Kane's equations." This is certainly not true, for Kane's method allows one to introduce generalized coordinates at the outset of any problem, just as with the Gibbs-Appell equations.

In the Note, the Gibbs function S was shown to have the following form:

$$S = \tfrac{1}{2} \sum_i \alpha_i \ddot{g}_i^2 \tag{1}$$

where the α_i are constants and the g_i are functions of the coordinates and their first and second derivatives. Desloge shows that it is never necessary to expand the squares indicated in Eq. (1), but that implicit differentiation of S can be used to form the dynamical equations. His claim is that one can construct S without carrying out the squaring operations, and then identify the α_i and g_i. His claim is certainly correct, but bears further examination. The quantities α_i and g_i can also be found by inspection from the expressions for the acceleration vectors for the various body mass centers that comprise the system. A small amount of labor can be saved by never forming Eq. (1) at all. Thus Gibbs' method can be improved by bypassing the formation of the Gibbs function. Since the steps leading to Eq. (1) were left out of the Note, the earliest point at which the α_i and g_i became "visible" was presumed to be Eq. (1), which is not true. Desloge goes on to state that there may be situations in which it is preferable to utilize a Gibbs function rather than Kane's method, but he has not offered a concrete example.

The analysis presented in the Note was concerned with the formation of nonlinear dynamical equations. Quite often, the analyst is actually interested in forming linearized equations, not by simplifying nonlinear equations but by directly forming linearized equations. In this connection, Kane's method affords the analyst a definite advantage over the use of a Gibbs function, because one is free to linearize at an earlier point in the analysis. A simple example will illustrate this point.

Point O moves on a fixed circle of radius R at a constant speed. A particle P of unit mass is connected to O by a massless rod of length l. Unit vector $\boldsymbol{b}_1$ is directed in the radial direction at O, while $\boldsymbol{b}_2$ is tangent to the circle at O. Linearized equations during which the angle q between the rod and $\boldsymbol{b}_1$ remain nearly zero are to be formed.

Received Feb. 27, 1987.

*Senior Scientist.